Introduction to Algebraic Geometry

B. Math (Hons) III year

Midterm

Time: 2 hours

Feb 28th 2020

Note: There are eight problems, each worth seven points. Your total score $x/56$ will be reported as $\min(2x, 100)\%$!

Remark: The base field in these problems is $\mathbb{C}$, and is often referred to as k. The polynomials rings associated with $\mathbb{A}^1$, $\mathbb{A}^2$, etc. will be written as $\mathbb{C}[x]$ (or $k[x]$), $\mathbb{C}[x,y]$ (or $k[x,y]$), etc.

1. The Plücker embedding in $\mathbb{P}^5$ of the Grassmanian $G(2,4)$ of 2-dimensional subspaces in $\mathbb{C}^4$ is determined by one equation. Derive that equation.

2. Prove that any nondegenerate conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$.

3. We know that for any two $n \times n$ matrices A and B over k, $Tr(AB) = Tr(BA)$, where Tr denotes the trace. Use a Zariski topology argument to show that, more generally, the product matrices AB and BA have the same characteristic polynomials (from which it follows that both matrices have the same trace!).

4. Let X_1 denote the set of $n \times m$ matrices over k of rank 1, considered as a subset of $\mathbf{P}^{nm-1}$. (Thus, we identify a matrix M of rank 1 with all its nonzero scalar multiples in this representation.) Show that X_1 is a closed subvariety of $\mathbf{P}^{nm-1}$, isomorphic to a product of suitable projective varieties of smaller dimension.

5. Show that in a Noetherian ring, there are only finitely many minimal prime ideals over an ideal I (that is, only finitely many prime ideals P such that $P \supseteq I$ and if Q is a prime ideal such that $P \supseteq Q \supset I$ then $P = Q$). Use this to show that if S/R is an integral extension of Noetherian rings, then over any prime ideal P of R, there are only finitely many prime ideals Q of S such that $Q \cap R = P$.

6. Let f be an irreducible polynomial in $k[x, y]$ of the form $f_n + f_{n-1}$, where f_n and f_{n-1} are nonzero homogenous polynomials of degree n and $n-1$ respectively (for some positive integer $n \geq 2$). Let C be the zero set of f in $\mathbb{A}^2$. Show that C can be parameterized by an open set of $\mathbb{A}^1$. Does your parameterization cover all points of C? Prove you assertions, of course.

7. Give examples (with proofs) of the following:

   (a) A morphism $f\colon X \mapsto Y$ of affine varieties such that $f(X)$ is neither open nor closed in Y but is dense in Y.

   (b) An isomorphism between projective varieties $X \subseteq \mathbb{P}^m$ and $Y \subseteq \mathbb{P}^n$ such that the corresponding homogenous coordinate rings are not isomorphic.

8. Let $C \subseteq \mathbb{A}^2$ consist of points of the form (t^4, t^6, t^9) as t varies over k. Show that C is an irreducible variety in $\mathbb{A}^2$. Determine its ideal. Show that C is not isomorphic to $\mathbb{A}^1$.